

Efficient Quantum Algorithm for Identifying Hidden Polynomials

Thomas Decker* Jan Draisma^{†‡} Pawel Wocjan[§]

September 2, 2008

Abstract

We consider a natural generalization of an abelian Hidden Subgroup Problem where the subgroups and their cosets correspond to graphs of linear functions over a finite field \mathbb{F} with d elements. The hidden functions of the generalized problem are not restricted to be linear but can also be m -variate polynomial functions of total degree $n \geq 2$.

The problem of identifying hidden m -variate polynomials of degree less or equal to n for fixed n and m is hard on a classical computer since $\Omega(\sqrt{d})$ black-box queries are required to guarantee a constant success probability. In contrast, we present a quantum algorithm that correctly identifies such hidden polynomials for all but a finite number of values of d with constant probability and that has a running time that is only polylogarithmic in d .

1 Introduction

Shor's algorithm for factoring integers and calculating discrete logarithms [21] is one of the most important and well known example of an exponential speed-up based on quantum computation. This algorithm as well as other fast quantum algorithms for number-theoretical problems [11, 12, 20, 16] essentially rely on the efficient solution of an abelian Hidden Subgroup Problem (HSP) [3]. This has naturally raised the questions of what interesting problems can be reduced to the non-abelian HSP and of whether the general non-abelian HSP can also be solved efficiently on a quantum computer.

It is known that an efficient quantum algorithm for the dihedral HSP would give rise to efficient quantum algorithms for certain lattice problems [19], and that an efficient

*School of Computer Science, McGill University, 3480 University Street, Montreal, Quebec H3A 2A7, Canada. Electronic address: decker@ira.uka.de

[†]Department of Mathematics and Computer Science, Technische Universiteit Eindhoven, PO Box 513, 5600 MB Eindhoven, The Netherlands. Electronic address: j.draisma@tue.nl

[‡]Centrum voor Wiskunde en Informatica, Amsterdam, The Netherlands.

[§]School of Electrical Engineering and Computer Science, University of Central Florida, Orlando, FL 32816, USA. Electronic address: wocjan@cs.ucf.edu

quantum algorithm for the symmetric group would give rise to an efficient quantum algorithm for the graph isomorphism problem [9]. Despite the fact that efficient algorithms have been developed for several non-abelian HSP's (see, for example, Ref. [15] and the references therein), the HSP over the dihedral group and the symmetric group have withstood all attempts so far. Moreover, there is evidence that the non-abelian HSP might be hard for some groups such as the symmetric group [14].

Another idea to generalize abelian HSP is to consider Hidden Shift Problems [4, 7] or problems with hidden non-linear structures [5, 13, 22]. In the latter context, we define and analyze a black-box problem that is based on polynomial functions of degree $n \geq 2$ and that can be reduced to an instance of the yet unsolved Hidden Polynomial Problem (HPP) [5]. Although our problem can be seen as a special case we refer to it as HPP in the following. The subgroups and the cosets of the HSP are generalized to graphs of polynomial multivariate functions going through the origin and to translated function graphs, respectively.

To solve this new problem, we use the “pretty good measurement” framework, which was introduced in Ref. [2] to obtain efficient quantum algorithms for the HSP over some semidirect product groups. First, we reduce the HPP to a quantum state identification problem. Second, we design a measurement scheme for distinguishing the states. Third, we relate the success probability and implementation to a classical algebro-geometric problem. The analysis of this classical problem leads us to an efficient quantum algorithm for the black-box problem.

This paper is organized as follows. In Section 2 we define the Hidden Polynomial Problem and show that it suffices to solve the univariate case on a quantum computer. In Section 3 we reduce this case to a state distinguishing problem and present a measurement scheme to solve it. In Section 4, we prove that the measurement scheme can be implemented efficiently and its success probability is bounded from below by a constant, which is independent of d . To do this, we analyze the properties of an algebro-geometric problem related to the black-box problem. In Section 5 we conclude and discuss possible objectives for further research.

2 Hidden Polynomial Problem

The Hidden Polynomial Problem is a natural generalization of the abelian HSP over groups of the special form $G := \mathbb{F}^{m+1}$. The hidden subgroup is defined by the m generators $(0, \dots, 1, \dots, 0, q_i) \in \mathbb{F}^{m+1}$ where the 1 is in the i th component and q_i is in \mathbb{F} . In this case, the hidden subgroup H_Q and its cosets $H_{Q,z}$ for $z \in \mathbb{F}$ are given by

$$H_Q := \{(x, Q(x)) : x \in \mathbb{F}^m\} \quad \text{and} \quad H_{Q,z} := \{(x, Q(x) + z) : x \in \mathbb{F}^m\}$$

where Q is the unknown linear polynomial $Q(X_1, \dots, X_m) = q_1 X_1 + \dots + q_m X_m$. For the HPP we also consider polynomials of higher degree.

Definition 2.1. Let \mathbb{F} be a finite field with d elements and characteristic p and let $Q(X_1, \dots, X_m) \in \mathbb{F}[X_1, \dots, X_m]$ be an arbitrary polynomial with total degree

$\deg(Q) \leq n$ and vanishing constant term¹. Furthermore, let $B : \mathbb{F}^{m+1} \rightarrow \mathbb{F}$ be a black-box function with

$$B(r_1, \dots, r_m, s) := \pi(s - Q(r_1, \dots, r_m))$$

where π is an unknown (but fixed) arbitrary permutation of the elements of \mathbb{F} . The Hidden Polynomial Problem is to identify the polynomial Q if only the black-box function B is given.

Remark 2.2 (General Definition of HPP). The general HPP, which is defined in Ref. [5], can be equivalently reformulated as follows: The black-box function $h : \mathbb{F}^\ell \rightarrow \mathbb{F}$ is given by $h(r_1, \dots, r_\ell) := \pi(P(r_1, \dots, r_\ell))$, where σ is an unknown (but fixed) arbitrary permutation of \mathbb{F} and $P(X_1, \dots, X_\ell)$ is the hidden polynomial. Hence, the black-boxes B from Def. 2.1 occur as special cases when the polynomials P are restricted to have the form

$$P(X_1, \dots, X_m, Y) := Y - Q(X_1, \dots, X_m).$$

This restriction makes it possible to obtain an efficient quantum algorithm.

Remark 2.3 (Classical Query Complexity). To derive a lower bound on the classical query complexity, we only consider the case of univariate polynomials of degree 1. Due to the permutation π the function values $B(r, s)$ themselves are useless. We need to obtain at least one collision, i.e., two different points (r, s) and (\tilde{r}, \tilde{s}) with $B(r, s) = B(\tilde{r}, \tilde{s})$, to determine the slope of the hidden line. Assume we have queried the black-box B at N different points and have not seen any collision. Then we can exclude at most $\binom{N}{2} = O(N^2)$ different slopes. Since there are d different slopes and all are equally likely, we have to make $\Omega(\sqrt{d})$ queries to determine the slope with constant success probability.

We say that a quantum algorithm for this problem is efficient if its running time is polylogarithmic in the field size d for a fixed number m of variables and a fixed maximum total degree n . We present such an efficient algorithm by first classically reducing the m -variate problem to the univariate problem and then by solving the univariate on a quantum computer. The reduction is described in the following lemma. For simplicity, we initially assume that the univariate case can be solved with probability 1 and show then how to deal with the other cases.

Lemma 2.4. *Assume that we can solve the univariate problem of degree n or less with success probability 1. Then, there is a simple recursive interpolation scheme that solves the m -variate problem by solving of at most*

$$\kappa_m = n^{m-1} + n^{m-2} + \dots + 1 \tag{1}$$

univariate problems.

Proof. First, rewrite Q as

$$Q(X_1, \dots, X_m) = \sum_{\alpha} Q_{\alpha}(X_m) \cdot X_1^{\alpha_1} \cdot \dots \cdot X_{m-1}^{\alpha_{m-1}}$$

¹A polynomial with constant term could also be considered in the following discussions. However, the constant term is randomized by our algorithm and cannot be determined as a consequence.

where $\alpha = (\alpha_1, \dots, \alpha_{m-1})$ is a vector with the exponents of the variables X_1, \dots, X_{m-1} . For the recursion we assume that we have an efficient algorithm for polynomials with $m - 1$ variables or less. Then we solve the m -variate problem with the following two steps.

- Step 1: Set the variables X_1, \dots, X_{m-1} to 0. We obtain

$$Q(0, \dots, 0, X_m) = Q_{(0, \dots, 0)}(X_m),$$

which is a univariate polynomial. It has no constant term because Q also has no constant term. This is a univariate problem and can be solved by assumption.

- Step 2: For n different fixed $t_j \in \mathbb{F}$ we consider² the polynomials

$$Q(X_1, \dots, X_{m-1}, t_j) = \sum_{\alpha} Q_{\alpha}(t_j) \cdot X_1^{\alpha_1} \cdot \dots \cdot X_{m-1}^{\alpha_{m-1}}$$

where $Q_{\alpha}(t_j)$ is a constant coefficient. By assumption we can determine all $Q_{\alpha}(t_j)$ for $\alpha \neq (0, \dots, 0)$. Denote by $|\alpha| = \sum_j \alpha_j$ the degree of the monomial defined by α . Since for $|\alpha| \geq 1$ the polynomial $Q_{\alpha}(X_m)$ has degree $n - |\alpha|$ and since we know n function values, we can determine Q_{α} efficiently with Lagrange interpolation [10].

Let κ_m be the total number of univariate problems with degree n or less that we have to solve in the recursive scheme. We have $\kappa_1 = 1$ and $\kappa_m = \kappa_1 + n \cdot \kappa_{m-1}$. This leads to the expression in Eq. (1). \square

We have assumed that the univariate case can be solved with success probability 1. However, our quantum algorithm fails to correctly identify the hidden univariate polynomial with some nonzero (but constant) probability p_f . We can reduce the failure probability of the quantum algorithm for the univariate case to p_f/κ_m by repeating it a certain number of times, which is independent of d . Then, by the union bound we see that the failure probability of the overall algorithm for the m -variate problem is at most p_f .

3 Distinguishing Polynomial Function States

Most quantum algorithms for HSP's are based on the standard approach, which reduces black-box problems to state distinguishing problems. We apply this approach to the Hidden Polynomial Problem as follows:

- Evaluate the black-box function on an equally weighted superposition of all $(r, s) \in \mathbb{F}^2$. The resulting state is

$$\frac{1}{d} \sum_{r, s \in \mathbb{F}} |r\rangle \otimes |s\rangle \otimes |\pi(s - Q(r))\rangle$$

²Note that the degree of each variable in the polynomials is w.l.o.g. smaller than the size d of \mathbb{F} after reducing exponents modulo $d - 1$, which is the order of the multiplicative group \mathbb{F}^{\times} .

- Measure and discard the third register. Assume we have obtained the result $\pi(z)$ with $z := s - Q(r)$. Then the state on the first and second register is $\rho_{Q,z} := |\phi_{Q,z}\rangle\langle\phi_{Q,z}|$ where

$$|\phi_{Q,z}\rangle := \frac{1}{\sqrt{d}} \sum_{r \in \mathbb{F}} |r\rangle \otimes |Q(r) + z\rangle$$

with the unknown polynomial Q , and z is uniformly at random. The corresponding density matrix is

$$\rho_Q := \frac{1}{d} \sum_{z \in \mathbb{F}} |\phi_{Q,z}\rangle\langle\phi_{Q,z}|. \quad (2)$$

We refer to the states ρ_Q as *polynomial function states*. We have to distinguish these states in order to solve the black-box problem.

3.1 Structure of Polynomial Function States

To obtain a compact expressions for polynomial function states ρ_Q we introduce the shift operator

$$S_\Delta := \sum_{x \in \mathbb{F}} |\Delta + x\rangle\langle x|$$

for $\Delta \in \mathbb{F}$, which directly leads to

$$\rho_Q = \frac{1}{d^2} \sum_{b,c \in \mathbb{F}} |b\rangle\langle c| \otimes S_{Q(b)-Q(c)}.$$

Now we use the fact that the shift operators S_Δ for all $\Delta \in \mathbb{F}$ can be diagonalized simultaneously with the Fourier transform

$$\text{DFT}_{\mathbb{F}} := \frac{1}{\sqrt{d}} \sum_{x,y \in \mathbb{F}} \omega_p^{\text{Tr}(xy)} |x\rangle\langle y|$$

over \mathbb{F} , where $\text{Tr} : \mathbb{F} \rightarrow \mathbb{F}_p$ is the trace map of the field extension \mathbb{F}/\mathbb{F}_p and $\omega_p := e^{2\pi i/p}$ is a primitive complex p th root of unity. The Fourier transform $\text{DFT}_{\mathbb{F}}$ can be approximated to within error ϵ in time polynomial in $\log(|\mathbb{F}|)$ and $\log(1/\epsilon)$ [7]. For simplicity, we assume that it can be implemented perfectly (as the error can be made exponentially small with polynomial resources only). We have

$$\text{DFT}_{\mathbb{F}} \cdot S_\Delta \cdot \text{DFT}_{\mathbb{F}}^\dagger = \sum_{x \in \mathbb{F}} \omega_p^{\text{Tr}(\Delta x)} |x\rangle\langle x|.$$

Consequently, the density matrices have the block diagonal form

$$\begin{aligned} \tilde{\rho}_Q &:= (I_d \otimes \text{DFT}_{\mathbb{F}}) \cdot \rho_Q \cdot (I_d \otimes \text{DFT}_{\mathbb{F}}^\dagger) \\ &= \frac{1}{d^2} \sum_{b,c,x \in \mathbb{F}} \chi([Q(b) - Q(c)]x) |b\rangle\langle c| \otimes |x\rangle\langle x| \end{aligned}$$

in the Fourier basis where we set $\chi(z) := \omega_p^{\text{Tr}(z)}$ for all $z \in \mathbb{F}$ and where I_d denotes the identity matrix of size d .

By repeating the standard approach k times for the same black-box function B , we obtain the density matrix $\tilde{\rho}_Q^{\otimes k}$. After rearranging the registers we can write

$$\begin{aligned}
\tilde{\rho}_Q^{\otimes k} &= \frac{1}{d^{2k}} \sum_{b,c,x \in \mathbb{F}^k} \chi \left(\sum_{j=1}^k [Q(b_j) - Q(c_j)] x_j \right) |b\rangle\langle c| \otimes |x\rangle\langle x| \\
&= \frac{1}{d^{2k}} \sum_{b,c,x \in \mathbb{F}^k} \chi \left(\sum_{j=1}^k \left[\sum_{i=1}^n q_i (b_j^i - c_j^i) \right] x_j \right) |b\rangle\langle c| \otimes |x\rangle\langle x| \\
&= \frac{1}{d^{2k}} \sum_{b,c,x \in \mathbb{F}^k} \chi \left(\sum_{i=1}^n q_i \left[\sum_{j=1}^k (b_j^i - c_j^i) x_j \right] \right) |b\rangle\langle c| \otimes |x\rangle\langle x| \\
&= \frac{1}{d^{2k}} \sum_{b,c,x \in \mathbb{F}^k} \chi \left(\langle q, (\Phi_n(b) - \Phi_n(c))x \rangle \right) |b\rangle\langle c| \otimes |x\rangle\langle x|,
\end{aligned}$$

where q , $\Phi_n(b)$, and $\Phi_n(c)$ are defined as follows:

- $q := (q_1, q_2, \dots, q_n)^T \in \mathbb{F}^n$ is the column vector whose entries are the coefficients of the hidden polynomial $Q(X) = \sum_{i=1}^n q_i X^i$
- $\Phi_n(b)$ is the $n \times k$ matrix

$$\Phi_n(b) := \sum_{i=1}^n \sum_{j=1}^k b_j^i |i\rangle\langle j| = \begin{pmatrix} b_1 & b_2 & \cdots & b_k \\ b_1^2 & b_2^2 & \cdots & b_k^2 \\ \vdots & \vdots & & \vdots \\ b_1^n & b_2^n & \cdots & b_k^n \end{pmatrix}$$

- $\langle \cdot, \cdot \rangle$ denotes the map $\mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}$ with $\langle v, w \rangle = v_1 w_1 + \cdots v_n w_n$ for $v, w \in \mathbb{F}^n$.

3.2 Algebro-Geometric Problem

We now show how to construct an orthogonal measurement for distinguishing the states $\tilde{\rho}_Q^{\otimes k}$ by applying and suitably modifying the “pretty good measurement” techniques developed in [1, 2, 4]. Both the success probability and the efficient implementation of our measurement are closely related to the following algebro-geometric problem: Consider the problem to determine all $b \in \mathbb{F}^k$ for given $x \in \mathbb{F}^k$ and $w \in \mathbb{F}^n$ such that $\Phi_n(b) \cdot x = w$, i.e.,

$$\begin{pmatrix} b_1 & b_2 & \cdots & b_k \\ b_1^2 & b_2^2 & \cdots & b_k^2 \\ \vdots & \vdots & & \vdots \\ b_1^n & b_2^n & \cdots & b_k^n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \quad (3)$$

We denote the set of solutions to these polynomial equations and its cardinality by

$$S_w^x := \{b \in \mathbb{F}^k : \Phi_n(b) \cdot x = w\} \quad \text{and} \quad \eta_w^x := |S_w^x|,$$

respectively. We also define the quantum states $|S_w^x\rangle$ to be the equally weighted superposition of all solutions

$$|S_w^x\rangle := \frac{1}{\sqrt{\eta_w^x}} \sum_{b \in S_w^x} |b\rangle$$

if $\eta_w^x > 0$ and $|S_w^x\rangle$ to be the zero vector otherwise. Using this notation we can write the state $\tilde{\rho}_Q^{\otimes k}$ as

$$\tilde{\rho}_Q^{\otimes k} = \frac{1}{d^{2k}} \sum_{x \in \mathbb{F}^k} \sum_{w, v \in \mathbb{F}^n} \chi(\langle q, w \rangle - \langle q, v \rangle) \sqrt{\eta_w^x \eta_v^x} |S_w^x\rangle \langle S_v^x| \otimes |x\rangle \langle x|. \quad (4)$$

3.3 Idealized Measurement for Identifying the States

We first consider an idealized situation to explain the intuition behind the measurement that we will use in the following sections to solve the HPP efficiently. Assume that there is an efficient implementation of the unitary transformation U_x that depends on x and that satisfies the equation

$$U_x |S_w^x\rangle = |w\rangle \quad (5)$$

for all (x, w) with $\eta_w^x > 0$. Then, there is an efficient measurement for identifying the polynomial states with success probability

$$\frac{1}{d^{2k+n}} \sum_{x \in \mathbb{F}^k} \left(\sum_{w \in \mathbb{F}^n} \sqrt{\eta_w^x} \right)^2. \quad (6)$$

For the proof, we observe that the block structure of the states $\tilde{\rho}_Q^{\otimes k}$ in Eq. (4) implies that we can measure the second register in the computational basis without any loss of information. The probability of obtaining a particular x is

$$\text{Tr} \left(\tilde{\rho}_Q^{\otimes k} (I_{d^k} \otimes |x\rangle \langle x|) \right) = \frac{1}{d^{2k}} \sum_{w \in \mathbb{F}^n} \eta_w^x = \frac{1}{d^k},$$

i.e., we have the uniform distribution, and the resulting reduced state is

$$\tilde{\rho}_Q^x := \frac{1}{d^k} \sum_{w, v \in \mathbb{F}^n} \chi(\langle q, w \rangle - \langle q, v \rangle) \sqrt{\eta_w^x \eta_v^x} |S_w^x\rangle \langle S_v^x|. \quad (7)$$

We now apply U_x to the state $\tilde{\rho}_Q^x$ of Eq. (7) and obtain

$$U_x \tilde{\rho}_Q^x U_x^\dagger = \frac{1}{d^k} \sum_{w, v \in \mathbb{F}^n} \chi(\langle q, w \rangle - \langle q, v \rangle) \sqrt{\eta_w^x \eta_v^x} |w\rangle \langle v|.$$

After having applied the transform U_x , we measure in the Fourier basis, i.e., we carry out the orthogonal measurement with respect to the states

$$|\psi_{Q'}\rangle := \frac{1}{\sqrt{d^n}} \sum_{w \in \mathbb{F}^n} \chi(\langle q', w \rangle) |w\rangle \quad (8)$$

where q' ranges over all tuples in \mathbb{F}^n . Simple computations show that the probability for the correct identification of the state $\tilde{\rho}_Q^x$ is

$$\langle \psi_Q | \tilde{\rho}_Q^x | \psi_Q \rangle = \frac{1}{d^{k+n}} \left(\sum_{w \in \mathbb{F}^n} \sqrt{\eta_w^x} \right)^2. \quad (9)$$

The probability of correctly identifying Q is obtained by averaging, i.e., summing the probabilities in Eq. (9) over all x and multiplying the sum by $1/d^k$. It is equal to the expression in Eq. (6). This completes the proof.

The problem with this idealized measurement is that there are pairs (x, w) where η_w^x is in the order of d . It is not clear how to implement the unitary U_x in Eq. (5) efficiently in these cases. In the next subsection we consider an approximate version V_x of U_x . This approximation guarantees that $U_x |S_w^x\rangle = V_x |S_w^x\rangle$ is satisfied for pairs (x, w) with $1 \leq \eta_w^x \leq D$ where D is some constant. We show that V_x can be implemented efficiently and that the resulting approximate measurement is good enough to identify the states with constant success probability.

3.4 Approximate Measurement

In this and the following sections we set $k = n$, i.e., the number k of copies equals the maximum degree n of the hidden polynomials. Furthermore, let D be some positive integer that depends on n but not on d , let $X_{\text{good}} \subseteq \mathbb{F}^n$ be some subset, and for $x \in X_{\text{good}}$ let W_{good}^x be some subset of $\{w \in \mathbb{F}^n \mid 1 \leq \eta_w^x \leq D\}$. The number D and the sets X_{good} and W_{good}^x will be determined later. We define the subset

$$B_{\text{good}}^x := \{b \in \mathbb{F}^n \mid \Phi_n(b) \cdot x = w \text{ for some } w \in W_{\text{good}}^x\} \quad (10)$$

for all $x \in X_{\text{good}}$.

Lemma 3.1. *Assume that there are efficient classical methods for testing membership in X_{good} and W_{good}^x and for enumerating the elements of S_w^x for given $x \in X_{\text{good}}$ and $w \in W_{\text{good}}^x$. Then there is an efficient approximate measurement for identifying the states with success probability bounded from below by*

$$\frac{1}{d^{3n}} \cdot |X_{\text{good}}| \cdot |W_{\text{good}}|^2, \quad (11)$$

where $|W_{\text{good}}| := \min_{x \in X_{\text{good}}} |W_{\text{good}}^x|^2$.

Remark 3.2. Note that the lower bound is a constant if $|X_{\text{good}}| = \Omega(d^n)$ and $|W_{\text{good}}| = \Omega(d^n)$. We analyze the algebro-geometric problem and show that all the above properties are satisfied and the cardinalities of the sets are sufficiently large.

Proof. Let us assume that we have obtained $x \in X_{\text{good}}$ in the first measurement step as described in Section 3.3. The probability of this event is $|X_{\text{good}}|/d^n$. We now discuss the approximate transformation V_x and the resulting success probability. Let P_{good} be the projector onto the subspace spanned by $|b\rangle$ for all $b \in B_{\text{good}}^x$. Clearly, the orthogonal

measurement defined by P_{good} can be carried out efficiently since membership in W_{good}^x can be tested efficiently. The probability to be in the “good” subspace is

$$\text{Tr}\left(P_{\text{good}} \tilde{\rho}_Q^x P_{\text{good}}\right) = \frac{|B_{\text{good}}^x|}{d^n}$$

and the resulting reduced density operator is

$$\tilde{\rho}_{Q,\text{good}}^x := \frac{1}{|B_{\text{good}}^x|} \sum_{w,v \in W_{\text{good}}^x} \chi\left(\langle q, w \rangle - \langle q, v \rangle\right) \sqrt{\eta_w^x \eta_v^x} |S_w^x\rangle \langle S_v^x|. \quad (12)$$

In the following we use the fact that for $x \in X_{\text{good}}$ and all $w \in W_{\text{good}}^x$ the cardinality η_w^x is bounded from above by D and that the elements of the sets S_w^x can be computed efficiently. In this case we have an efficiently computable bijection between S_w^x and the set $\{(w, j) : j = 0, \dots, \eta_w^x - 1\}$. This bijection is obtained by sorting the elements of S_w^x according to the lexicographic order on \mathbb{F}^n and associating to each $b \in S_w^x$ the unique $j \in \{0, \dots, \eta_w^x - 1\}$ corresponding to its position in S_w^x .

We now show how to implement the transformation V_x efficiently, which satisfies

$$V_x |S_w^x\rangle = |w\rangle.$$

- Implement a transformation with

$$|b\rangle \otimes |0\rangle \otimes |0\rangle \mapsto |w\rangle \otimes |j\rangle \otimes |\eta_w^x\rangle \quad (13)$$

for all $b \in B_{\text{good}}^x$. To make it unitary we can simply map all $b \notin B_{\text{good}}^x$ onto some vectors that are orthogonal (e.g., by simply flipping some additional qubit saying that they are bad). Note that b and x determine j and w uniquely and vice versa. Furthermore, we can compute w and j efficiently since η_w^x is bounded from above by D . Consequently, this unitary acts on the states $|S_w^x\rangle$ as follows

$$\frac{1}{\sqrt{\eta_w^x}} \sum_{b \in S_w^x} |b\rangle \otimes |0\rangle \otimes |0\rangle \mapsto \frac{1}{\sqrt{\eta_w^x}} |w\rangle \otimes \sum_{j=1}^{\eta_w^x} |j\rangle \otimes |\eta_w^x\rangle \quad (14)$$

- Apply the unitary

$$\sum_{\ell=0}^{\eta_w^x-1} (F_{\ell+1} \oplus I_{d^n-\ell-1}) \otimes |\ell\rangle \langle \ell| + \sum_{\ell=\eta_w^x}^{d^n-1} I_{d^n} \otimes |\ell\rangle \langle \ell|$$

on the second and third register. This implements the embedded Fourier transform F_ℓ of size ℓ controlled by the second register in order to map the superposition of all $|j\rangle$ with $j \in \{0, \dots, \ell - 1\}$ to $|0\rangle$. The resulting state is $|w\rangle \otimes |0\rangle \otimes |\eta_w^x\rangle$.

- Uncompute $|\eta_w^x\rangle$ in the third register with the help of w and x . This leads to the state $|w\rangle \otimes |0\rangle \otimes |0\rangle$

We apply V_x to the state of Eq. (12) and obtain

$$V_x \tilde{\rho}_{Q,\text{good}}^x V_x^\dagger = \frac{1}{|B_{\text{good}}^x|} \sum_{w,v \in W_{\text{good}}^x} \chi(\langle q, w \rangle - \langle q, v \rangle) \sqrt{\eta_w^x \eta_v^x} |w\rangle \langle v|.$$

We now measure in the Fourier basis, i.e., we carry out the orthogonal measurement with respect to the states $|\psi_{Q'}\rangle$ defined in Eq. (8). Analogously to the ideal situation we obtain that the probability for the correct detection of the state $\tilde{\rho}_Q^x$ is

$$\langle \psi_Q | V_x \tilde{\rho}_{Q,\text{good}}^x V_x^\dagger | \psi_Q \rangle = \frac{1}{d^n} \frac{1}{|B_{\text{good}}^x|} \left(\sum_{w \in W_{\text{good}}^x} \sqrt{\eta_w^x} \right)^2. \quad (15)$$

The overall success probability is

$$\frac{1}{d^n} \sum_{x \in X_{\text{good}}} \frac{|B_{\text{good}}^x|}{d^n} \langle \psi_Q | V_x \tilde{\rho}_{Q,\text{good}}^x V_x^\dagger | \psi_Q \rangle = \frac{1}{d^{3n}} \sum_{x \in X_{\text{good}}} \left(\sum_{w \in W_{\text{good}}^x} \sqrt{\eta_w^x} \right)^2. \quad (16)$$

The first factor $1/d^n$ is the probability that we obtain a specific x . The right most expression is clearly at least the expression in Eq. (11). \square

4 Analysis of the Algebro-Geometric Problem

In this section we show that the cardinalities of the sets X_{good} and W_{good}^x in Lemma 3.1 are sufficiently large in the case $k = n$ for all \mathbb{F} that satisfy certain constraints on the characteristic. This guarantees that the success probability of the approximate measurement in Section 3.4 is bounded from below by a constant that does not depend on the field size.

Although our classical algebro-geometric problem appears to be very similar to the average-case problem in Ref. [2] for the HSP over semidirect product groups, the elementary arguments of Lemma 5 in Ref. [2] cannot be applied in a straightforward way to prove that the cardinalities of the sets X_{good} and W_{good}^x in Lemma 3.1 are sufficiently large. More precisely, in the case of the HPP we obtain the first two moments

$$\mathbb{E}[\eta_w^x] = d^{k-n} \text{ and} \quad (17)$$

$$\mathbb{E}[(\eta_w^x)^2] = \mathbb{E}[\eta_w^x] + \frac{1}{d^{k+n}} \sum_{b \neq c} \sum_{x \in \mathbb{F}^k} \delta[(\Phi_n(b) - \Phi_n(c))x = (0, 0, \dots, 0)^T] \quad (18)$$

for the η_w^x . Since we have $b \neq c$, there is an index j' with $b_{j'} \neq c_{j'}$. It is clear that for all b_j, c_j , and x_j with $j \neq j'$ we have at most one $x_{j'}$ such that the condition in the square bracket is satisfied but it is not obvious when this $x_{j'}$ exists. In contrast to the situation in Ref. [2], this argument only leads to a weak upper bound

$$\mathbb{E}[(\eta_w^x)^2] \leq \mathbb{E}[\eta_w^x] + \frac{1}{d^n} (d^k - 1) d^{k-1} \quad (19)$$

on the second moment. Eq. (17) implies that the number of copies should be at least n . In this case, however, the upper bound on the second moment is $\Omega(d^{n-1})$. Therefore, we cannot use the probabilistic arguments of Ref. [2] to prove that X_{good} and W_{good}^x have the desired properties.

In the following, we choose an approach that does not rely on any probabilistic arguments. We present two different proofs based on algebro-geometric techniques that also show that the approximative measurement can be implemented efficiently. Both proofs differ slightly in their scope: The first analysis applies if the characteristic of \mathbb{F} is larger than $k = n$ and the second if a certain polynomial with integer coefficients does not vanish when considered modulo the characteristic. Hence, the second analysis can be used in some cases when the first analysis cannot be applied and vice versa.

The notions and results of algebra and algebraic geometry that are used in the proofs can be found in Ref. [17] as well as in Refs. [6, 10, 18].

4.1 First Analysis

For the analysis of the implementation of V_x and the success probability of our algorithm for $k = n$ we define the n polynomials $f_j \in \mathbb{F}[X_1, \dots, X_n, B_1, \dots, B_n]$ as

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} := \begin{pmatrix} B_1 & B_2 & \cdots & B_n \\ B_1^2 & B_2^2 & \cdots & B_n^2 \\ \vdots & \vdots & & \vdots \\ B_1^n & B_2^n & \cdots & B_n^n \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix},$$

where the product of the matrix and the vector corresponds to the left-hand side of Eq. (3). Furthermore, let f be the n -tuple $f := (f_1, \dots, f_n)$, which defines a map from $\mathbb{F}^n \times \mathbb{F}^n$ to \mathbb{F}^n with $f(x, b) = (f_1(x, b), \dots, f_n(x, b))$. Using this notation, S_w^x can be expressed as

$$S_w^x = \{b \in \mathbb{F}^n : f(x, b) = w\} \quad \text{with} \quad w \in \mathbb{F}^n.$$

For a fixed x the tuple f defines a map from \mathbb{F}^n to \mathbb{F}^n and the sets S_w^x are the preimages of $w \in \mathbb{F}^n$ under this map.

Let $\overline{\mathbb{F}}$ denote the algebraic closure of \mathbb{F} . We also view f as a map from $\overline{\mathbb{F}}^n$ to $\overline{\mathbb{F}}^n$. For given $x, w \in \overline{\mathbb{F}}^n$, we refer to the subvariety $\{b \in \overline{\mathbb{F}}^n \mid f(x, b) = w\}$ of $\overline{\mathbb{F}}^n$ as the fiber of $f(x, \cdot)$ over w . In the proposition below, we choose the sets X_{good} and W_{good}^x such that the fibers of $f(x, \cdot)$ over w are zero-dimensional. This implies that the numbers η_w^x are bounded from above by some constant D for all $x \in X_{\text{good}}$ and $w \in W_{\text{good}}^x$ since the sets S_w^x are equal to the intersections of the fibers with \mathbb{F}^n .

Proposition 4.1. Assume that the characteristic p of \mathbb{F} is strictly larger than n , let $X_{\text{good}} := (\mathbb{F}^\times)^n$, and for $x \in X_{\text{good}}$ set

$$W_{\text{good}}^x := \{w \in \mathbb{F}^n \mid \text{the fiber of } f(x, \cdot) \text{ over } w \text{ is zero-dimensional and } \eta_w^x \geq 1\}.$$

Then the requirements of Lemma 3.1 are satisfied and we have $|X_{\text{good}}| = \Omega(d^n)$ and $|W_{\text{good}}^x| = \Omega(d^n)$.

Proof. We find the solutions of the system $f(x, b) = w$ efficiently as follows: We precompute generic reduced Gröbner bases with Buchberger's algorithm for the lexicographic order [10, 6], i.e., we treat the coefficients of the polynomials in the variables b_i as rational expressions in the variables x_i and w_i . Whenever Buchberger's algorithm requires division by a rational expression E in the x_i and w_i , we distinguish between the case where E remains nonzero upon specializing x and w and the case where E becomes zero upon specialization. This precomputation yields a finite decision tree whose leaves correspond to all possible reduced Gröbner bases. In each leaf we can decide whether the solution variety of the system $f(x, b) = w$ is zero-dimensional, and if so we can compute an upper bound on its cardinality. Choose D to be the maximum over all these upper bounds.

On input (\mathbb{F}, x, w) we now find the corresponding Gröbner basis by evaluating a bounded number of rational expressions that also only needs a bounded number of field operations. From the Gröbner basis we can read off whether the set of solutions, i.e., the fiber of $f(x, \cdot)$ over w is zero-dimensional. If this is the case, the set S_w^x of all solutions $b \in \mathbb{F}^n$ can be computed by iteratively solving a bounded number of univariate equations, which again can be done efficiently. By construction, this set has cardinality at most D .

We now show that $|W_{\text{good}}^x| = \Omega(d^n)$ for all $x \in X_{\text{good}}$. Fix $x \in X_{\text{good}}$. On the open set \hat{U} in $\overline{\mathbb{F}}^n$ where all coordinates b_i are distinct, the differential $d\varphi$ of the map $\varphi : \hat{U} \rightarrow \overline{\mathbb{F}}^n$ sending b to $f(x, b)$ has full rank everywhere. Indeed, at b the differential of this map sends $c \in \overline{\mathbb{F}}^n$ to

$$\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & \ddots \\ & & & & n \end{pmatrix} \begin{pmatrix} 1 & \dots & 1 \\ b_1 & \dots & b_n \\ b_1^2 & \dots & b_n^2 \\ \vdots & & \vdots \\ b_1^{n-1} & \dots & b_n^{n-1} \end{pmatrix} \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Now the first matrix is invertible because the characteristic of \mathbb{F} is larger than n , and the second matrix is invertible because the b_i are distinct. Hence if $d|_b \varphi$ maps c to 0 then all $c_i x_i$ are zero, and as $x \in (\mathbb{F}^\times)^n$ we find $c = 0$, i.e., $d|_b \varphi$ is injective.

This implies that the fibers of φ over w are all zero-dimensional.³ Their cardinalities are bounded from above by D . Let U denote the intersection of \hat{U} with \mathbb{F}^n . The upper bound implies that the size of the image $\varphi(U)$ is at least $|\varphi(U)| \geq |U|/D = \Omega(d^n)$. Clearly, the fibers of $f(x, \cdot)$ over w are zero-dimensional for all $w \in \varphi(U)$ that do not lie in the image of the complement of \hat{U} under the map $f(x, \cdot)$. This image is certainly contained in some subvariety $\hat{I}_x \subseteq \overline{\mathbb{F}}^n$ defined over \mathbb{F} of dimension $n - 1$ since $\dim(\overline{\mathbb{F}}^n \setminus \hat{U}) = n - 1$. Hence, we can apply Schwartz-Zippel's theorem (Prop. 98 in Ref. [23]) and conclude that the cardinality of the intersection I_x of \hat{I}_x with \mathbb{F}^n is at most κd^{n-1} . Here κ is a uniform upper bound on the degree of the equation defining I_x ,

³This is an elementary statement from algebraic geometry: If some fiber has positive dimension, then it contains a point b where the tangent space to the fiber has positive dimension. This tangent space is then mapped to zero by $d|_b \varphi$, a contradiction to the injectivity of this linear map. For a concise introduction to the interplay between dimension and tangent spaces we refer to [6, chapter 9, paragraph 6].

which can again be found by a generic Gröbner basis computation without specifying x . This completes the proof that for each $x \in X_{\text{good}}$ the number of w such that the fiber of $f(x, \cdot)$ over w is zero-dimensional is $\Omega(d^n)$. \square

With Lemma 3.1 the following corollary is a direct consequence of Prop. 4.1.

Corollary 4.2. *For $p > n$ the approximative measurement of Sec. 3.4 can be implemented efficiently. Furthermore, for the success probability we have*

$$\begin{aligned} \frac{1}{d^{3n}} \sum_{x \in \mathbb{F}^n} \left(\sum_{w \in \mathbb{F}^n} \sqrt{\eta_w^x} \right)^2 &\geq \frac{1}{d^{3n}} \sum_{x \in (\mathbb{F}^\times)^n} \left(\sum_{w \in \varphi(U) \setminus I_x} \sqrt{\eta_w^x} \right)^2 \\ &\geq \frac{1}{d^{3n}} (d-1)^n \left(\frac{d(d-1) \cdots (d-n+1)}{D} - \kappa d^{n-1} \right)^2 \\ &= 1/D^2 - O(1/d), \end{aligned}$$

which leads to a lower bound that does not depend on the field size d .

4.2 Second Analysis

The following general proposition allows us to make statements about the size of the preimages of a general morphism $f : \mathbb{A}^m \times \mathbb{A}^n \rightarrow \mathbb{A}^n$ over an affine space \mathbb{A} independently of the underlying field \mathbb{F} . This morphism should be thought of as a family of morphisms from the n -dimensional space \mathbb{A}^n to itself, parameterized by \mathbb{A}^m .

Proposition 4.3. Consider a morphism $f : \mathbb{A}^m \times \mathbb{A}^n \rightarrow \mathbb{A}^n$ over \mathbb{Z} , that is, f is given by an n -tuple $f = (f_1, \dots, f_n)$ of polynomials in $\mathbb{Z}[X, B]$, where $X = (X_1, \dots, X_m)$ and $B = (B_1, \dots, B_n)$ are the coordinates on \mathbb{A}^m and on the first copy of \mathbb{A}^n , respectively. Suppose that the Jacobian determinant $\det(\partial f_i / \partial B_j)_{ij}$ is a non-zero element⁴ of $\mathbb{Z}[X, B]$. Then there exists a real number γ with $0 < \gamma \leq 1$ and a non-zero polynomial $g \in \mathbb{Z}[X]$ such that for all finite fields \mathbb{F} and all $x \in \mathbb{F}^m$ with $g(x) \neq 0$ when g is considered as a polynomial over \mathbb{F} we have $|f(\{x\} \times \mathbb{F}^n)| \geq \gamma |\mathbb{F}|^n$.

Proof. By the condition on the Jacobian determinant $f_1, \dots, f_n \in \mathbb{Q}(X, B)$ are algebraically independent over $\mathbb{Q}(X)$.⁵ As $\mathbb{Q}(X, B)$ has transcendence degree n over $\mathbb{Q}(X)$, every B_i is algebraic over $\mathbb{Q}(X, f_1, \dots, f_n)$, i.e., there exist non-zero polynomials $P_1, \dots, P_n \in \mathbb{Z}[X, W, T]$ such that $P_i(X, f, B_i) = 0 \in \mathbb{Z}[X, B]$. View P_i as a polynomial of degree $d_i \in \mathbb{N}$ in T with coefficients from $\mathbb{Z}[X, W]$, and let $Q_i \in \mathbb{Z}[X, W]$ be the (non-zero) coefficient of T^{d_i} in P_i . Then $h := \prod_{i=1}^n Q_i(X, W)$ is a non-zero

⁴This condition on f says that generic morphisms in this family are dominant. When we work over algebraically closed fields \mathbb{F} this means that the image is dense in \mathbb{F}^n . The proposition states that over finite fields the generic morphism still hits a large subset of \mathbb{F}^n .

⁵If $P \in \mathbb{Q}(X)[W_1, \dots, W_n]$ is of minimal degree with $P(f) = P(f_1, \dots, f_n) = 0$, then differentiation with respect to B_j and the chain rules gives $\sum_i \frac{\partial P}{\partial W_i}(f) \frac{\partial f_i}{\partial B_j} = 0$, so that $(\frac{\partial P}{\partial W_i}(f))_i$ is in the row kernel of the Jacobian matrix, and non-zero by minimality of $\deg(P)$ —whence $\det(\frac{\partial f_i}{\partial B_j}) = 0$.

polynomial in $\mathbb{Z}[X, W]$. By the algebraic independence of the f_i , $h(X, f(X, B))$ is a non-zero polynomial in $\mathbb{Z}[X, B]$; viewing this as a polynomial of degree e in B with coefficients from $\mathbb{Z}[X]$, let $g \in \mathbb{Z}[X]$ be any non-zero coefficient of a monomial B^α of degree e .

Now let \mathbb{F} be any finite field and let $x \in \mathbb{F}^m$ be such that $g(x) \neq 0$. Then $q := h(x, f(x, B))$ is a non-zero polynomial in $\mathbb{F}[B]$ of degree e . For any $b \in \mathbb{F}^n$ outside the zero set of q we have $Q_i(x, f(x, b)) \neq 0$ so that $P_i(x, f(x, b), T) \in \mathbb{F}[T]$ has degree d_i , for all $i = 1, \dots, n$. Again by construction, any $b' \in \mathbb{F}^n$ satisfying $f(x, b') = f(x, b)$ satisfies the system of polynomial equations $P_i(x, f(x, b), b'_i) = 0$ for $i = 1, \dots, n$, which has at most $D := \prod_i d_i$ solutions. We conclude that the fiber of $f(x, \cdot)$ over $f(x, b)$ has a cardinality of at most D , and therefore

$$|f(\{x\} \times \mathbb{F}^n)| \geq \frac{|\{b \in \mathbb{F}^n \mid q(b) \neq 0\}|}{D}$$

The Schwartz-Zippel theorem applied to q shows that the right-hand side of this inequality is at least $(|\mathbb{F}|^n - e|\mathbb{F}|^{n-1})/D$. From this the existence of γ follows. \square

Remark 4.4. The polynomials P_i, g , and h can all be computed effectively, e.g., using Gröbner basis methods [10, 6]. In general, the running time will depend very strongly on the particular form of the morphism f , but it is independent of the field size d , which is sufficient for our purposes. It is possible that a more refined analysis taking into account the structure of f could lead to an improved performance for certain types of morphisms.

Remark 4.5. We emphasize that we cannot rule out that the polynomial $g \in \mathbb{Z}[X]$ is zero when considered as a polynomial over \mathbb{F} . This can only happen if all coefficients of g are multiples of the characteristic p of \mathbb{F} . For this reason, we have to exclude all finite fields with these characteristics.

Proposition 4.6. Let the f_i be as in Subsection 4.1 and g as in Prop. 4.3. Assume that the polynomial g is non-zero when considered over the finite field \mathbb{F} . Furthermore, define the set

$$X_{\text{good}} := \{x \in \mathbb{F}^n \mid g(x) \neq 0\}$$

and for $x \in X_{\text{good}}$ the set

$$W_{\text{good}}^x := \{w \in \mathbb{F}^n \mid h(x, w) \neq 0 \text{ and } \eta_w^x \geq 1\},$$

where $h \in \mathbb{Z}[X, W]$ is the polynomial from the proof of Prop. 4.3. Furthermore, take the constant D as in the proof. Then Lemma 3.1 can be applied. In particular, the approximative measurement of Sec. 3.4 can be implemented efficiently and its success probability is bounded from below by a positive and non-zero constant independent of d .

Proof. In our application of Prop. 4.3 we have $m = n$ and the Jacobian determinant $\det(\partial f_i / \partial B_j)$ is non-zero as after specializing all X_i to 1 it is a non-zero scalar times the Vandermonde determinant $\det(B_j^{i-1})_{ij}$. This shows that we have a non-zero Jacobian matrix. If the image of g in $\mathbb{F}[X]$ is non-zero then by the Schwartz-Zippel theorem

at least $|\mathbb{F}|^n - \deg(g) \cdot |\mathbb{F}|^{n-1}$ of the elements $x \in \mathbb{F}^m$ lie in X_{good} , hence we have $|X_{\text{good}}| \in O(d^n)$. By the proof of Prop. 4.3, for all $x \in X_{\text{good}}$ the set B_{good}^x from Eq. (10) contains $O(d^n)$ elements $b \in \mathbb{F}^n$ with $q(b) \neq 0$. Since for these b the fiber of $f(x, \cdot)$ over $f(x, b)$ contains at most D elements, we also have $O(d^n)$ elements in W_{good}^x . With Rem. 3.2 the lower bound for the success probability follows.

The membership in X_{good} can be computed efficiently because we only have to evaluate $g(x)$. Furthermore, for given $x \in X_{\text{good}}$ and $w \in \mathbb{F}^n$ the membership of w in W_{good}^x can be checked efficiently: By computing the zeros of the univariate polynomials $P_i(x, w, T)$ in \mathbb{F} we find the possible values for each of the b_i , and then we need only to determine⁶ those combinations that are mapped to w . This also allows us to compute S_w^x efficiently for $x \in X_{\text{good}}$ and $w \in W_{\text{good}}^x$. \square

Using these results, we show that the success probability of the approximate measurement is bounded from below by a constant for $n = 2$ and fields of characteristic $p = 2$. Recall that the first analysis cannot be applied in these cases since the characteristic is not strictly greater than the degree.

Example 4.7. We consider the case $n = 2$ and find the two polynomials

$$\begin{aligned} P_1(X_1, X_2, W_1, W_2, T) &:= (-X_1X_2 - X_1^2)T^2 + (2W_2X_1)T + (W_1X_2 - W_2^2) \\ P_2(X_1, X_2, W_1, W_2, T) &:= (-X_1X_2 - X_2^2)T^2 + (2W_2X_2)T + (W_1X_1 - W_2^2) \end{aligned}$$

with the leading terms

$$\begin{aligned} Q_1(X_1, X_2, W_1, W_2) &:= -X_1X_2 - X_1^2 \\ Q_2(X_1, X_2, W_1, W_2) &:= -X_1X_2 - X_2^2. \end{aligned}$$

Therefore, we have

$$h(X_1, X_2, W_1, W_2) = X_1X_2(X_1 + X_2)^2,$$

i.e., the polynomial $h \in \mathbb{Z}[X, W]$ is of degree zero in W and we have

$$g(X_1, X_2) = X_1X_2(X_1 + X_2)^2.$$

Hence, for the maximum degree $n = 2$ of the hidden functions we find polynomials P_1 and P_2 where $x \in \mathbb{F}^2$ with $g(x) \neq 0$ exists for all finite fields \mathbb{F} with $|\mathbb{F}| \geq 3$.

5 Conclusion and Outlook

We have shown that certain instances of the hidden polynomial problem that are hard on classical computers can be solved efficiently on a quantum computer for a fixed total degree n and a fixed number m of indeterminates provided that the characteristic of the underlying field meets certain constraints.

⁶This can be done more efficiently by the replacement of the P_i with a triangular system that can be used to find the elements of S_w^x consecutively.

The extension of our results to arbitrary characteristics p of the field \mathbb{F} , to more general algebraic structures, e.g., rings with Fourier transforms, and the extension to a broader class of functions such as rational functions are possible objectives of future research. Additionally, it would be important to find other polynomial black-boxes with efficient quantum algorithms and to explore if interesting real-life problems can be reduced efficiently to such black-box problems.

Acknowledgments

T.D. was supported by CIFAR, NSERC, QuantumWorks, MITACS and the ARO/NSA quantum algorithms grant W911NSF-06-1-0379. J.D. was supported by DIAMANT, a mathematics cluster funded by NWO, the Netherlands Organisation for Scientific Research. P.W. gratefully acknowledges the support by NSF grants CCF-0726771 and CCF-0746600.

References

- [1] D. Bacon, A. Childs, and W. van Dam, *Optimal measurements for the dihedral hidden subgroup problem*, Chicago Journal of Theoretical Computer Science, article 2, 2006.
- [2] D. Bacon, A. Childs, and W. van Dam, *From optimal measurements to efficient quantum algorithms for the hidden subgroup problem over semidirect product groups*, Proc. of the 46th Annual Symposium on Foundations of Computer Science, pp. 469-478, 2005.
- [3] R. Boneh and R. Lipton, *Quantum cryptanalysis of hidden linear functions*, Proc. Advances in Cryptology, Lecture Notes in Computer Science, vol. 963, pp. 424-437, 1995.
- [4] A. Childs and W. van Dam, *Quantum algorithm for a generalized hidden shift problem*, Proc. of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 1225-1234, 2007.
- [5] A. Childs, L. Schulman, and U. Vazirani, *Quantum algorithms for hidden nonlinear structures*, Proc. of the 48th Annual Symposium on Foundations of Computer Science, pp. 395-404, 2007, arXiv: 0705.2784v1
- [6] D. Cox, J. Little, and D. O’Shea, *Ideals, varieties, and algorithms – An introduction to computational algebraic geometry and commutative algebra*, Springer, 1997.
- [7] W. van Dam, S. Hallgren, and L. Ip, *Quantum Algorithms for some Hidden Shift Problems*, SIAM Journal on Computing, vol. 36, no. 3, pp. 763-778, 2006.
- [8] T. Decker and P. Wocjan, *Efficient quantum algorithm for hidden quadratic and cubic polynomial function graphs*, arXiv: quant-ph/0703195v3
- [9] M. Ettinger and P. Høyer, *A quantum observable for the graph isomorphism problem*, arXiv: quant-ph/9901029

- [10] J. von zur Gathen and J. Gerhard, *Modern Computer Algebra*, Cambridge University Press, 2003.
- [11] S. Hallgren, *Polynomial-time quantum algorithms for Pell's equation and the principal ideal problem*, Proc. 34th ACM Symposium on Theory of Computing, pp. 653–658, 2002.
- [12] S. Hallgren, *Fast quantum algorithms for computing the unit group and class group of a number field*, Proc. 37th ACM Symposium on Theory of Computing, pp. 468–474, 2005.
- [13] S. Hallgren, A. Russell, and I. Shparlinski, *Quantum noisy rational function reconstruction*, Lecture Notes in Computer Science, vol. 3595, pp. 420–429, 2005.
- [14] S. Hallgren, C. Moore, M. Rötteler, A. Russell, and P. Sen, *Limitations of quantum coset states for graph isomorphism*, Proc. of 38th ACM Symposium on Theory of Computing, pp. 604 – 617, 2006.
- [15] G. Ivanyos, L. Sanselme, and M. Santha, *Quantum algorithm for the hidden subgroup problem in extraspecial groups*, Proc. of 24th Annual Symposium on Theoretical Aspects of Computer Science, Lecture Notes in Computer Science, vol. 4393, pp. 586–597, 2007.
- [16] K. Kedlaya, *Quantum computation of zeta functions of curves*, Computational Complexity, vol. 15, issue 1, pp. 1–9, 2006.
- [17] S. Lang, *Algebra*, Graduate Texts in Mathematics 211, Springer, 2002.
- [18] J. Harris, *Algebraic Geometry: A First Course*, Graduate Texts in Mathematics 133, Springer, 1995.
- [19] O. Regev, *Quantum computation and lattice problems*, Proc. 43rd Symposium on Foundations of Computer Science, pp. 520–529, 2002.
- [20] A. Schmidt and U. Vollmer, *Polynomial time quantum algorithm for the computation of the unit group of a number field*, Proc. 37th ACM Symposium on Theory of Computing, pp. 475–480, 2005.
- [21] P. Shor, *Polynomial-time algorithms for prime factorizations and discrete logarithms on a quantum computer*, SIAM Journal on Computing, vol. 26, pp. 1484–1509, 1997.
- [22] I. Shparlinski and A. Russell, *Classical and quantum polynomial reconstruction via Legendre symbol evaluation*, Journal of Complexity, vol. 20, no. 2-3, pp. 404–422, 2004.
- [23] R. Zippel, *Effective polynomial computation*, Kluwer, 1993.